

Finite Element Methods for Interface Problems: Robust and Local Optimal A Priori Error Estimates

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Abstract. For elliptic interface problems in two- and three-dimension, this paper establishes a priori error estimates for the Crouzeix-Raviart nonconforming, the Raviart-Thomas mixed, and the discontinuous Galerkin finite element approximations. These estimates are robust with respect to the diffusion coefficient and optimal with respect to local regularity of the solution. Moreover, we obtain these estimates with no assumption on the distribution of the diffusion coefficient.

1 Introduction

As a prototype of problems with interface singularities, this paper studies *a priori* error estimates of various finite element methods for the following interface problem (i.e., the diffusion problem with discontinuous coefficients):

$$-\nabla \cdot (\alpha(x) \nabla u) = f \quad \text{in } \Omega \quad (1.1)$$

with homogeneous Dirichlet boundary conditions (for simplicity)

$$u = 0 \quad \text{on } \partial\Omega, \quad (1.2)$$

where Ω is a bounded polygonal domain in \mathbb{R}^d with $d = 2$ or 3 ; $f \in L^2(\Omega)$ is a given function; and diffusion coefficient $\alpha(x)$ is positive and piecewise constant with possible large jumps across subdomain boundaries (interfaces):

$$\alpha(x) = \alpha_i > 0 \quad \text{in } \Omega_i \quad \text{for } i = 1, \dots, n.$$

Here, $\{\Omega_i\}_{i=1}^n$ is a partition of the domain Ω with Ω_i being an open polygonal domain. The variational formulation for the interface problem in (1.1) and (1.2) is to find $u \in H_0^1(\Omega)$ such that

$$(\alpha \nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \quad (1.3)$$

It is well known that the solution u of problem (1.3) belongs to $H^{1+s}(\Omega)$ with possibly very small $s > 0$.

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Let $\mathcal{T} = \{K\}$ be a regular triangulation of the domain Ω (see, e.g., [14, 5]). Denote by h_K the diameter of the element K . Assume that interfaces $\{\partial\Omega_i \cap \partial\Omega_j : i, j = 1, \dots, n\}$ do not cut through any element $K \in \mathcal{T}$. For any element $K \in \mathcal{T}$, denote by $P_k(K)$ the space of polynomials on K with total degree less than or equal to k . Denote the continuous finite element space on the triangulation \mathcal{T} by

$$V_k^c = \{v \in H_0^1(\Omega) : v|_K \in P_k(K) \forall K \in \mathcal{T}\}.$$

Then the conforming finite element method is to find $u_k^c \in V_k^c$ such that

$$(\alpha \nabla u_k^c, \nabla v) = (f, v) \quad \forall v \in V_k^c. \quad (1.4)$$

The following a priori error estimate was established in [3]:

$$\|\alpha^{1/2} \nabla(u - u_k^c)\|_{0,\Omega} = \inf_{v \in V_k^c} \|\alpha^{1/2} \nabla(u - v)\|_{0,\Omega} \leq C \left(\sum_{K \in \mathcal{T}} h_K^{2s} \alpha_K \|\nabla u\|_{s,K}^2 \right)^{1/2}. \quad (1.5)$$

Here and thereafter, we use C with or without subscripts to denote a generic positive constant that is independent of the mesh parameter and the jump of $\alpha(x)$ but that may depend on the domain Ω . The estimate in (1.5) is robust with respect to α , but not optimal with respect to the local regularity since s is a global exponent. This kind of a priori error estimate is not satisfactory. For example, for the well-known Kellogg's example of the interface problem in [20, 8], the solution of the underlying problem has low regularity on elements along the physical interfaces, but is very smooth on elements away from the physical interfaces.

By Sobolev's embedding theorem (see, e.g., [18]), $H^{1+s}(\Omega)$, with $s > 0$ for the two-dimension and $s > 1/2$ for the three-dimension, is embedded in $C^0(\Omega)$ and, hence, the nodal interpolation of the solution u is well-defined. In [16], it is proved that if $v \in H^{1+s}(K)$ with $s > 0$ in the two-dimension, then for $0 < t \leq s$, the following estimate holds for the linear nodal interpolation I_K :

$$\|v - I_K v\|_{0,K} \leq C h^{1+t} |\nabla v|_{t,K}.$$

With the same technique, we can also prove the result for $s > 1/2$ in the three-dimension. This implies the following a priori error estimate that is not only robust with respect to the jump of α but also with respect to the local regularity (see Section 3.3 of [23] in the two-dimension).

Corollary 1.1. *Let $u \in H^{1+s}(\Omega)$ with $s > 0$ be the solution of problem (1.5). Assume that $s > 1/2$ for $d = 3$ and that the restriction of u on element K belongs to $H^{1+s_K}(K)$ for all $K \in \mathcal{T}$. Then*

$$\|\alpha^{1/2} \nabla(u - u_k^c)\|_{0,\Omega} \leq C \left(\sum_{K \in \mathcal{T}} h_K^{2\min\{k, s_K\}} \alpha_K |\nabla u|_{s_K, K}^2 \right)^{1/2}. \quad (1.6)$$

Remark 1.2. *In the case that $s \in (0, 1/2]$ in the three-dimension, under the quasi-monotonicity assumption (QMA) on the distribution of the coefficient $\alpha(x)$ (see section 1.1), estimate (1.6) may be obtained through comparison results with discontinuous Galerkin method or Clément-type of interpolations (see [13]).*

The a priori error estimate using local regularity in (1.6) is the base for adaptive finite element methods to achieve equal discretization error distribution (see [22] for examples in

both the one- and two-dimensions) and, hence, is important. Moreover, the QMA is already restrictive enough in the two-dimension and is much worse in the three-dimension. The purpose of this paper is to derive estimates of this type for the Crouzeix-Raviart nonconforming, the Raviart-Thomas mixed, and the discontinuous Galerkin finite element approximations. These estimates hold when the solution of (1.3) has low global regularity, i.e., $s \in (0, 1/2]$ in the three-dimension, and, in particular, the distribution of the coefficient does not satisfy the QMA. In this sense, these estimates are better than that of the conforming finite element methods. Analysis for the mixed elements is rather straightforward. However, derivation of such estimates for the nonconforming and the discontinuous elements is non-trivial. In order to achieve them, we prove the robust Céa's Lemma type of results for the Crouzeix-Raviart nonconforming and the discontinuous Galerkin finite element approximations for the first time. Besides making use of both analytical approaches developed recently in the respective [7] and [19], we also need to establish new trace inequalities (see Lemmas 2.3 and 2.4). These trace inequalities also play an important role in the a posteriori error estimates (see [12]).

Standard a priori error estimate for the discontinuous elements (see, e.g., [1, 26]) requires the underlying problem being sufficiently smooth, i.e, at least piecewise $H^{3/2+\epsilon}$, so that there is an error equation. For problems with low regularity, by carefully defining duality pairs on element interfaces, in [7] we developed a non-standard variational formulation that, in term, leads to an error equation and then an a priori error estimate. The estimate in [7] is robust with respect to α without the QMA, but not local optimal due to the use of a continuous approximation in our analysis. An alternative approach was developed by Gudi [19] for the Poisson equation. His approach compares the discontinuous solution with the continuous solution, and makes use of the efficiency bound of the a posteriori error estimation. Moreover, it is applicable to problems with low regularity. Its application to interface problems with the Oswald analyzed introduced in [7] would yield an a priori error estimate that is robust under the QMA.

The paper is organized as follows. Section 2 introduces Sobolev spaces of fractional order and establishes some new trace inequalities that play an important role in both the a priori and a posteriori error estimates. Various finite element approximations are described in section 3. Robust and local optimal a priori error estimates without QMA are derived in section 4.

1.1 Quasi-Monotonicity Assumption

To establish the a priori and, in particular, the a posteriori error estimates to be robust with respect to the diffusion coefficient $\alpha(x)$, one often requires its distribution satisfying certain conditions. Hypothesis 2.7 in [3] is a monotonic condition that is weaken to Quasi-Monotonicity Assumption (QMA) in [24]. Such a condition also appeared in the convergence analysis of the domain decomposition method in [15].

Quasi-Monotonicity Assumption. *Assume that any two different subdomains $\bar{\Omega}_i$ and $\bar{\Omega}_j$, which share at least one point, have a connected path passing from $\bar{\Omega}_i$ to $\bar{\Omega}_j$ through adjacent subdomains such that the diffusion coefficient $\alpha(x)$ is monotone along this path.*

This assumption is needed in all previous papers on the robustness of the interface problem, e.g., [3, 7, 8, 9, 11, 24]. Robust estimates established in this paper do not require the QMA.

2 Sobolev Space and Preliminaries

2.1 Sobolev space of fractional order

Let Ω be a non-empty open set in \mathbb{R}^d . We use the standard notation and definitions for the Sobolev spaces $H^m(\Omega)^d$ and $H^m(\partial\Omega)^d$ with integer $m \geq 0$; the standard associated inner products are denoted by $(\cdot, \cdot)_{m,\Omega}$ and $(\cdot, \cdot)_{m,\partial\Omega}$, and their respective norms (semi-norms) are denoted by $\|\cdot\|_{m,\Omega}$ and $\|\cdot\|_{m,\partial\Omega}$ ($|\cdot|_{m,\Omega}$ and $|\cdot|_{m,\partial\Omega}$). We suppress the superscript d because their dependence on dimension will be clear by context. We also omit the subscript Ω from the inner product and norm designation when there is no risk of confusion. For $m = 0$, $H^m(\Omega)^d$ coincides with $L^2(\Omega)^d$. In this case, the inner product and norm will be denoted by $\|\cdot\|_0$ and (\cdot, \cdot) , respectively.

For $t \in (0, 1)$, the following semi-norm

$$|v|_{t,\Omega} = \left(\int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^2}{|x - y|^{d+2t}} dx dy \right)^{1/2} \quad 0 < t < 1$$

is used to define Sobolev spaces of fractional order. For integer $m \geq 0$, Sobolev space $H^s(\Omega)$ with $s = m + t$ is equipped with the norm

$$\|v\|_{s,\Omega} = \left(\sum_{|\alpha| \leq m} \int_{\Omega} |\partial^{\alpha} v|^2 dx + |v|_{s,\Omega}^2 \right)^{1/2}, \quad (2.1)$$

where $|v|_{s,\Omega}$ is a semi-norm defined by

$$|v|_{s,\Omega} = \left(\sum_{|\alpha|=m} |\partial^{\alpha} v|_{t,\Omega}^2 \right)^{1/2}. \quad (2.2)$$

Sobolev spaces with negative indecies are defined through duality.

Another way to define Sobolev spaces of fractional order is by the method of interpolation. To this end, let $B_1 \subset B_0$ be Banach spaces. For $t > 0$ and $u \in B_0$, define the K -functional by

$$K(t, u) = \inf_{v \in B_1} (\|u - v\|_{B_0}^2 + t^2 \|v\|_{B_1}^2)^{1/2}.$$

For $0 < \theta < 1$, the interpolation space $B_{\theta} = [B_0, B_1]_{\theta}$ is a Banach space equipped with the norm

$$\|u\|_{[B_0, B_1]_{\theta}} = N_{\theta} \left(\int_0^{\infty} |t^{-\theta} K(t, u)|^2 \frac{dt}{t} \right)^{1/2}, \quad (2.3)$$

where $N_{\theta} > 0$ is a normalization factor.

For any real numbers $s_0 \leq s_1$, let $s = m + t = (1 - \theta)s_0 + \theta s_1$ with $\theta \in (0, 1)$. It was shown (see Theorem B.8 in [21]) that

$$[H^{s_0}(\Omega), H^{s_1}(\Omega)]_{\theta} = H^s(\Omega)$$

and that the norms defined in (2.1) and (2.3) are identical if the normalization factor is chosen to be $N_{\theta} = \sqrt{\frac{2 \sin(\pi\theta)}{\pi}}$. Moreover, for $v \in H^{s_1}(\Omega)$, it was shown (see Theorem B.1 in [21]) that

$$\|v\|_{s,\Omega} \leq \sqrt{\frac{\sin(\pi\theta)}{\pi\theta(1-\theta)}} \|v\|_{s_0,\Omega}^{1-\theta} \|v\|_{s_1,\Omega}^{\theta}. \quad (2.4)$$

Lemma 2.1. *Let $s > 0$, $t \in [0, s)$, and $K \in \mathcal{T}$. Assume that v is a given function in $H^s(K)$. For any given $\epsilon > 0$, there exists a small $\delta \in (0, s - t)$, depending on v , such that*

$$\|v\|_{t+\delta, K} \leq (1 + \epsilon) \|v\|_{t, K}. \quad (2.5)$$

Proof. Obviously, (2.5) holds for $v = 0$. Assume that $v \neq 0$. For any $\delta \in (0, s - t)$, we have

$$H^{t+\delta}(K) = [H^t(K), H^s(K)]_\theta \quad \text{with } \theta = \frac{\delta}{s - t},$$

which, together with (2.4), implies

$$\|v\|_{t+\delta, K} \leq \sqrt{\frac{\sin(\pi\theta)}{\pi\theta(1-\theta)}} \|v\|_{t, K}^{1-\theta} \|v\|_{s, K}^\theta = \sqrt{\frac{\sin(\pi\theta)}{\pi\theta(1-\theta)}} \left(\frac{\|v\|_{s, K}}{\|v\|_{t, K}} \right)^\theta \|v\|_{t, K}.$$

Now, (2.5) is a consequence of the fact that

$$\lim_{\theta \rightarrow 0} \sqrt{\frac{\sin(\pi\theta)}{\pi\theta(1-\theta)}} \left(\frac{\|v\|_{s, K}}{\|v\|_{t, K}} \right)^\theta = 1.$$

This completes the proof of the lemma. \square

Remark 2.2. *Since $\|v\|_{t, K} \leq \|v\|_{t+\delta, K}$, Lemma 2.1 implies that*

$$\lim_{\delta \rightarrow 0^+} \|v\|_{t+\delta} = \|v\|_t.$$

Note that this continuity is not uniform with respect to v .

2.2 trace inequalities

For any $K \in \mathcal{T}$ and some $\alpha > 0$, let

$$V^{1+\alpha}(K) = \{v \in H^{1+\alpha}(K) : \Delta v \in L^2(K)\}.$$

Lemma 2.3. *Let F be a face of $K \in \mathcal{T}$ and let $s > 0$. Assume that v is a given function in $V^{1+s}(K)$. Then there exists a small $0 < \delta < \min\{s, 1/2\}$, depending on v , and a positive constant C independent of δ such that*

$$\|\nabla v \cdot \mathbf{n}\|_{\delta-1/2, F} \leq C (\|\nabla v\|_{0, K} + h_K \|\Delta v\|_{0, K}). \quad (2.6)$$

Proof. For any $v \in V^{1+s}(K)$, it was shown in [2, 7] that for all $0 < \delta < \min\{s, 1/2\}$, we have

$$\|\nabla v \cdot \mathbf{n}\|_{\delta-1/2, F} \leq C \left(\|\nabla v\|_{\delta, K} + h_K^{1-\delta} \|\Delta v\|_{0, K} \right),$$

which, together with Lemma 2.1 with $t = 0$ and the fact that $h_K^{-\delta} \leq 2$ for sufficiently small δ , implies the validity of (2.6). This completes the proof of the lemma. \square

Lemma 2.4. *Let F be a face of $K \in \mathcal{T}$, \mathbf{n}_F the unit vector normal to F , and $s > 0$. Assume that v is a given function in $V^{1+s}(K)$. For any $w_h \in P_k(K)$, we have*

$$\begin{aligned} \int_F (\nabla v \cdot \mathbf{n}_F) w_h ds &\leq C h_F^{-1/2} \|w_h\|_{0, F} (\|\nabla v\|_{0, K} + h_K \|\Delta v\|_{0, K}) \\ &\leq C h_K^{-1} \|w_h\|_{0, K} (\|\nabla v\|_{0, K} + h_K \|\Delta v\|_{0, K}). \end{aligned} \quad (2.7)$$

Proof. The second inequality in (2.7) follows from the inverse inequality. To show the validity of the first inequality in (2.7), as discussed in [7], $\int_F (\nabla v \cdot \mathbf{n}_F) w_h ds$ may be viewed as a duality pairing between $H^{\delta-1/2}(F)$ and $H^{1/2-\delta}(F)$ for all $0 < \delta < \min\{s, 1/2\}$. It follows from the definition of the dual norm, the inverse inequality, and (2.6) for sufficiently small δ that

$$\begin{aligned} \int_F (\nabla v \cdot \mathbf{n}_F) w_h ds &\leq \|\nabla v \cdot \mathbf{n}\|_{\delta-1/2, F} \|w_h\|_{1/2-\delta, F} \\ &\leq C h_K^{\delta-1/2} \|w_h\|_{0, F} (\|\nabla v\|_{0, K} + h_K \|\Delta v\|_{0, K}) \\ &\leq C h_K^{-1/2} \|w_h\|_{0, F} (\|\nabla v\|_{0, K} + h_K \|\Delta v\|_{0, K}). \end{aligned}$$

This completes the proof of the first inequality in (2.7) and, hence, the lemma. \square

Remark 2.5. Generalizations of the above results to $\boldsymbol{\tau} \in \{\boldsymbol{\tau} \in H^\alpha(K)^s : \nabla \cdot \boldsymbol{\tau} \in L^2(K)\}$ are obvious.

3 Various Finite Element Methods

Let \mathcal{N} be the set of vertices of the triangulation \mathcal{T} and \mathcal{N}_D be the collection of the vertices on the Dirichlet boundary. Denote by \mathcal{E}_K the set of faces of element $K \in \mathcal{T}$. In this paper, face means edge/face in the two-/three-dimension. Denote the set of all faces of the triangulation \mathcal{T} by

$$\mathcal{E} := \mathcal{E}_I \cup \mathcal{E}_D,$$

where \mathcal{E}_I and \mathcal{E}_D are the respective sets of all interior and boundary faces. For each $F \in \mathcal{E}$, denote by h_F the diameter of the face F and by \mathbf{n}_F a unit vector normal to F . For each interior face $F \in \mathcal{E}_I$, let K_F^- and K_F^+ be the two elements sharing the common face F such that the unit outward normal vector of K_F^- coincides with \mathbf{n}_F . When $F \in \mathcal{E}_D$, \mathbf{n}_F is the unit outward normal vector of $\partial\Omega$ and denote the element by K_F^- . For any $F \in \mathcal{E}$, denote by $v|_F^-$ and $v|_F^+$, respectively, the traces of a function v over F . Define jumps over faces by

$$[[v]]_F := \begin{cases} v|_F^- - v|_F^+ & F \in \mathcal{E}_I, \\ v|_F^- & F \in \mathcal{E}_D. \end{cases}$$

Denote the continuous and discontinuous finite element spaces on the triangulation \mathcal{T} by $V_k^c = \{v \in H_0^1(\Omega) : v|_K \in P_k(K) \forall K \in \mathcal{T}\}$ and $D_k = \{v \in L^2(\Omega) : v|_K \in P_k(K) \forall K \in \mathcal{T}\}$, respectively. Denote the Crouzeix-Raviart linear nonconforming finite element space by

$$V^{cr} = \{v \in L^2(\Omega) : v|_K \in P_1(K) \forall K \in \mathcal{T}, \int_F [[v]] ds = 0, \forall F \in \mathcal{E}\}$$

and the $H(\text{div})$ conforming Raviart-Thomas finite element space by

$$RT_k = \{\boldsymbol{\tau} \in H(\text{div}; \Omega) : \boldsymbol{\tau}|_K \in P_k(K)^d + \mathbf{x}P_k(K) \forall K \in \mathcal{T}\}.$$

Denote by ∇_h the discrete gradient operator that is defined element-wisely. Then the nonconforming finite element method is to find $u^{cr} \in V^{cr}$ such that

$$(\alpha \nabla_h u^{cr}, \nabla_h v) = (f, v) \quad \forall v \in V^{cr}. \quad (3.1)$$

3.1 mixed finite element method

Introducing the flux

$$\boldsymbol{\sigma} = -\alpha(x)\nabla u,$$

the mixed variational formulation for the problem in (1.1) and (1.2) is to find $(\boldsymbol{\sigma}, u) \in H(\text{div}; \Omega) \times L^2(\Omega)$ such that

$$\begin{cases} (\alpha^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau}) - (\nabla \cdot \boldsymbol{\tau}, u) = 0 & \forall \boldsymbol{\tau} \in H(\text{div}; \Omega), \\ (\nabla \cdot \boldsymbol{\sigma}, v) = (f, v) & \forall v \in L^2(\Omega). \end{cases} \quad (3.2)$$

Then the mixed finite element method is to find $(\boldsymbol{\sigma}_k^m, u_k^m) \in RT_k \times D_k$ such that

$$\begin{cases} (\alpha^{-1}\boldsymbol{\sigma}_k^m, \boldsymbol{\tau}) - (\nabla \cdot \boldsymbol{\tau}, u_k^m) = 0 & \forall \boldsymbol{\tau} \in RT_k, \\ (\nabla \cdot \boldsymbol{\sigma}_k^m, v) = (f, v) & \forall v \in D_k. \end{cases} \quad (3.3)$$

Difference between (3.2) and (3.3) yields the following error equation:

$$\begin{cases} (\alpha^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_k^m), \boldsymbol{\tau}) - (\nabla \cdot \boldsymbol{\tau}, u - u_k^m) = 0 & \forall \boldsymbol{\tau} \in RT_k, \\ (\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_k^m), v) = 0 & \forall v \in D_k. \end{cases} \quad (3.4)$$

Let f_k be the L^2 projection of f onto D_k for $k \geq 0$. Define local and global weighted oscillations by

$$\text{osc}_\alpha(f, K) = \frac{h_K}{\sqrt{\alpha_K}} \|f - f_{k-1}\|_{0,K} \quad \text{and} \quad \text{osc}_\alpha(f, \mathcal{T}) = \left(\sum_{K \in \mathcal{T}} \text{osc}_\alpha(f, K)^2 \right)^{1/2},$$

respectively.

3.2 discontinuous Galerkin finite element method

To describe discontinuous Galerkin finite element method, we need to introduce extra notations. To this end, let ω_F^+ and ω_F^- be weights defined on F satisfying $w_F^+(x) + w_F^-(x) = 1$, and introduce the following weighted averages

$$\{v(x)\}_w^F = \begin{cases} w_F^- v_F^- + w_F^+ v_F^+ & F \in \mathcal{E}_I, \\ v|_F^- & F \in \mathcal{E}_D \end{cases} \quad \text{and} \quad \{v(x)\}_F^w = \begin{cases} w_F^+ v_F^- + w_F^- v_F^+ & F \in \mathcal{E}_I, \\ 0 & F \in \mathcal{E}_D \end{cases}$$

for all $F \in \mathcal{E}$. A simple calculation leads to the following identity:

$$\llbracket uv \rrbracket_F = \{v\}_F^w \llbracket u \rrbracket_F + \{u\}_F^w \llbracket v \rrbracket_F. \quad (3.5)$$

For any $F \in \mathcal{E}_I$, denote by α_F^+ and α_F^- the diffusion coefficients on K_F^+ and K_F^- , respectively. Denote the arithmetic and the harmonic averages of α on $F \in \mathcal{E}$ by

$$\alpha_{F,A} = \begin{cases} \frac{\alpha_F^+ + \alpha_F^-}{2} & F \in \mathcal{E}_I, \\ \alpha_F^- & F \in \mathcal{E}_D \end{cases} \quad \text{and} \quad \alpha_{F,H} = \begin{cases} \frac{2\alpha_F^+ \alpha_F^-}{\alpha_F^+ + \alpha_F^-}, & F \in \mathcal{E}_I, \\ \alpha_F^- & F \in \mathcal{E}_D, \end{cases}$$

respectively, which are equivalent to the respective maximum and minimum of α :

$$\frac{1}{2} \max\{\alpha_F^+, \alpha_F^-\} \leq \alpha_{F,A} \leq \max\{\alpha_F^+, \alpha_F^-\} \quad \text{and} \quad \min\{\alpha_F^+, \alpha_F^-\} \leq \alpha_{F,H} \leq \frac{1}{2} \min\{\alpha_F^+, \alpha_F^-\}. \quad (3.6)$$

For $s > 0$, let

$$\begin{aligned} H^{1+s}(\mathcal{T}) &= \{v \in L^2(\Omega) : v|_K \in H^{1+s}(K) \ \forall K \in \mathcal{T}\} \\ \text{and } V^{1+s}(\mathcal{T}) &= \{v \in H^{1+s}(K) : (\Delta v)|_K \in L^2(K) \ \forall K \in \mathcal{T}\}. \end{aligned}$$

In [7] we introduced the following variational formulation for the interface problem in (1.1) and (1.2): find $u \in V^{1+\epsilon}(\mathcal{T})$ with $\epsilon > 0$ such that

$$a_{dg}(u, v) = (f, v) \quad \forall v \in V^{1+\epsilon}(\mathcal{T}), \quad (3.7)$$

where the bilinear form $a_{dg}(\cdot, \cdot)$ is given by

$$\begin{aligned} a_{dg}(u, v) &= (\alpha \nabla_h u, \nabla_h v) + \sum_{F \in \mathcal{E}} \int_F \gamma \frac{\alpha_{F,H}}{h_F} \llbracket u \rrbracket \llbracket v \rrbracket ds \\ &\quad - \sum_{F \in \mathcal{E}} \int_F \{\alpha \nabla u \cdot \mathbf{n}_F\}_w^F \llbracket v \rrbracket ds - \sum_{F \in \mathcal{E}} \int_F \{\alpha \nabla v \cdot \mathbf{n}_F\}_w^F \llbracket u \rrbracket ds. \end{aligned}$$

The γ is a positive constant only depending on the shape of elements. In order to guarantee robust error estimate with respect to α , we choose the following harmonic weights:

$$w_F^\pm = \frac{\alpha_F^\mp}{\alpha_F^- + \alpha_F^+}. \quad (3.8)$$

The discontinuous Galerkin finite element method is then to seek $u_k^{dg} \in D_k$ such that

$$a_{dg}(u_k^{dg}, v) = (f, v) \quad \forall v \in D_k. \quad (3.9)$$

Difference between (3.7) and (3.9) leads to the following error equation

$$a_{dg}(u - u_k^{dg}, v) = 0 \quad \forall v \in D_k. \quad (3.10)$$

For simplicity, we consider only this symmetric version of the interior penalty discontinuous Galerkin finite element method since its extension to other versions of discontinuous Galerkin approximations is straightforward. Define the jump semi-norm and the DG norm by

$$\|v\|_{J,F} = \sqrt{\frac{\alpha_{F,H}}{h_F}} \|\llbracket v \rrbracket\|_{0,F} \quad \text{and} \quad \|v\|_{dg} = \left(\|\alpha^{1/2} \nabla_h v\|_{0,\Omega}^2 + \sum_{F \in \mathcal{E}} \|v\|_{J,F}^2 \right)^{1/2},$$

respectively, for all $v \in H^1(\mathcal{T})$. It was shown in [7] that there exists a positive constant C independent of the jump of α such that

$$C \|v\|_{dg}^2 \leq a_{dg}(v, v) \quad \forall v \in D_k. \quad (3.11)$$

4 Robust and Local Optimal A Priori Error Estimates

4.1 CR nonconforming finite element method

Let

$$\begin{aligned} W^{1,1}(\mathcal{T}) &= \{v \in L^2(\Omega) : v|_K \in W^{1,1}(K) \ \forall K \in \mathcal{T}\} \\ \text{and } W(\mathcal{T}) &= \{v \in W^{1,1}(\mathcal{T}) : \int_F \llbracket v \rrbracket ds = 0 \ \forall F \in \mathcal{E}\}. \end{aligned}$$

Denote by $\theta_F(\mathbf{x})$ the nodal basis function of V^{cr} associated with the face $F \in \mathcal{E}$, i.e.,

$$\frac{1}{|F'|} \int_{F'} \theta_F(\mathbf{x}) ds = \delta_{FF'} \ \forall F' \in \mathcal{E},$$

where $\delta_{FF'}$ is the Kronecker delta. The local and global Crouzeix-Raviart interpolants are defined respectively by

$$I_K^{cr} v = \sum_{F \in \mathcal{E} \cap \partial K} \left(\frac{1}{|F|} \int_F v ds \right) \theta_F(\mathbf{x}) \quad \text{and} \quad I^{cr} v = \sum_{F \in \mathcal{E}} \left(\frac{1}{|F|} \int_F v ds \right) \theta_F(\mathbf{x})$$

for the respective $v \in W^{1,1}(K)$ and $v \in W(\mathcal{T})$. It was shown (see, e.g., Theorem 1.103 and Example 1.106 (ii) of [17]) that for $v \in H^{1+t}(K)$ with $0 \leq t \leq 1$

$$\|v - I_K^{cr} v\|_{0,K} \leq C h_K^{1+t} |\nabla v|_{t,K}. \quad (4.1)$$

Theorem 4.1. *Let u be the solution of (1.3) and u_K be its restriction on $K \in \mathcal{T}$. Assume that $u \in H^{1+s}(\Omega) \cap V^{1+s}(\mathcal{T})$ for some $s > 0$ and that $u|_K \in H^{1+s_K}(K)$ with element-wise defined $s_K > 0$ for all $K \in \mathcal{T}$. Let $u^{cr} \in V^{cr}$ be the nonconforming finite element approximation in (3.1). For both the two- and three-dimension, the following error estimates,*

$$\begin{aligned} \|\alpha^{1/2} \nabla_h(u - u^{cr})\|_0 &\leq C \left(\inf_{v \in V^{cr}} \|\alpha^{1/2} \nabla_h(u - v)\|_0 + \text{osc}_\alpha(f, \mathcal{T}) \right) \\ &\leq C \left(\left(\sum_{K \in \mathcal{T}} h_K^{2 \min\{1, s_K\}} |\alpha^{1/2} \nabla u|_{s_K, K}^2 \right)^{1/2} + \text{osc}_\alpha(f, \mathcal{T}) \right) \end{aligned} \quad (4.2)$$

hold, where C is a positive constant independent of the jump of the diffusion coefficient α .

Proof. The second inequality in (4.2) is an immediate consequence of (4.1). By Strang's lemma, to show the validity of the first inequality in (4.2), it suffices to prove

$$\sup_{w \in V^{cr}} \frac{|(f, w) - (\alpha \nabla u, \nabla_h w)|}{\|\alpha^{1/2} \nabla_h w\|_{0,\Omega}} \leq C \left(\inf_{v \in V^{cr}} \|\alpha^{1/2} \nabla_h(u - v)\|_{0,\Omega} + \text{osc}_\alpha(f, \mathcal{T}) \right). \quad (4.3)$$

To this end, for any $w \in V^{cr}$ and any $F \in \mathcal{E}$, by the fact that $\int_F \llbracket w \rrbracket ds = 0$, the mean value of w over F is single-valued constant, i.e.,

$$\bar{w}_F = \frac{1}{|F|} \int_F w|_{K_F^+} ds = \frac{1}{|F|} \int_F w|_{K_F^-} ds,$$

where K_F^+ and K_F^- are two elements sharing the common face F . Moreover, $\bar{w}_F = 0$ for $F \in \mathcal{E}_D$. Hence, by the continuity of the flux $\mathbf{n} \cdot \alpha \nabla u$ across face $F \in \mathcal{E}_I$, we have

$$\sum_{K \in \mathcal{T}} \sum_{F \in \partial K} \int_F (\mathbf{n} \cdot \alpha \nabla u) \bar{w}_F ds = \sum_{F \in \mathcal{E}} \int_F [(\mathbf{n} \cdot \alpha \nabla u) \bar{w}_F] ds = 0. \quad (4.4)$$

Now, it follows from (3.1), integration by parts, (4.4), the fact that $(\mathbf{n}_F \cdot \alpha \nabla v|_K)_F$ is a constant, and (2.7) that for all $v \in V^{cr}$

$$\begin{aligned} (\alpha \nabla u, \nabla_h w) - (f, w) &= \sum_{K \in \mathcal{T}} \int_{\partial K} (\mathbf{n} \cdot \alpha \nabla u) w ds = \sum_{K \in \mathcal{T}} \sum_{F \in \partial K} \int_F (\mathbf{n} \cdot \alpha \nabla u) (w - \bar{w}_F) ds \\ &= \sum_{K \in \mathcal{T}} \sum_{F \in \partial K} \int_F (\mathbf{n} \cdot \alpha \nabla (u - v)) (w - \bar{w}_F) ds \\ &\leq C \sum_{K \in \mathcal{T}} \sum_{F \in \partial K} h_K^{-1/2} \|w - \bar{w}_F\|_{0,F} (\|\alpha \nabla (u - v)\|_{0,K} + h_K \|f\|_{0,K}) \\ &\leq C \sum_{K \in \mathcal{T}} \sum_{F \in \partial K} \left(\|\alpha^{1/2} \nabla (u - v)\|_{0,K} + h_K \alpha_K^{-1/2} \|f\|_{0,K} \right) \|\alpha^{1/2} \nabla w\|_{0,K}. \end{aligned}$$

The last inequality is due to the fact that $\|w - \bar{w}_F\|_{0,F} \leq C h_K^{1/2} \|\nabla w\|_{0,K}$. Now, the Cauchy-Schwarz inequality gives

$$\frac{|(\alpha \nabla u, \nabla_h w) - (f, w)|}{\|\alpha^{1/2} \nabla_h w\|_{0,\Omega}} \leq C \left(\inf_{v \in V^{cr}} \|\alpha^{1/2} \nabla_h (u - v)\|_0 + \left(\sum_{K \in \mathcal{T}} h_K^2 \alpha_K^{-1} \|f\|_{0,K}^2 \right)^{1/2} \right)$$

for all $w \in V^{cr}$. Without QMA, in a similar fashion as the proof of the efficiency bound for the residual error estimator of discontinuous Galerkin finite element method (Lemma 5.2 of [7]), we have

$$h_K \alpha_K^{-1/2} \|f\|_{0,K} \leq C \left(\|\alpha^{1/2} \nabla_h (u - v)\|_{0,\omega_K} + \text{osc}_\alpha(f, K) \right)$$

for all $v \in V^{cr}$ and all $K \in \mathcal{T}$. Combining the above two inequalities implies the validity of (4.3). This completes the proof of the theorem. \square

Since linear conforming finite element solution $u_1^c \in V^{cr}$, we have

$$\inf_{v \in V^{cr}} \|\alpha^{1/2} \nabla_h (u - v)\|_0 \leq \|\alpha^{1/2} \nabla_h (u - u_1^c)\|_{0,\Omega},$$

which, together with Theorem 4.1, implies the following robust comparison result between linear conforming finite element and Crouzeix-Raviart nonconforming finite element approximations.

Corollary 4.2. *Without QMA, there exists a positive constant C independent of the jump of the diffusion coefficient such that*

$$\|\alpha^{1/2} \nabla_h (u - u^{cr})\|_{0,\Omega} \leq C \left(\|\alpha^{1/2} \nabla_h (u - u_1^c)\|_{0,\Omega} + \text{osc}_\alpha(f, \mathcal{T}) \right).$$

4.2 RT mixed finite element method

For fixed $s > 0$, denote by $I_k^{rt} : H(\text{div}; \Omega) \cap [H^s(\Omega)]^d \mapsto RT_k$ the standard RT interpolation operator satisfying the following approximation property: for $\boldsymbol{\tau} \in H^{s_K}(K)$

$$\|\boldsymbol{\tau} - I_k^{rt} \boldsymbol{\tau}\|_{0,K} \leq Ch_K^{\min\{s_K, k+1\}} |\boldsymbol{\tau}|_{s_K, K} \quad \forall K \in \mathcal{T}. \quad (4.5)$$

(The estimate in (4.5) is standard for $s_K \geq 1$ and may be proved by the average Taylor series developed in [16] and the standard reference element technique with Piola transformation for $0 < s_K < 1$.) Denote by $Q_k : L^2(\Omega) \mapsto D_k$ the L^2 -projection onto D_k . The following commutativity property is well-known:

$$\nabla \cdot (I_k^{rt} \boldsymbol{\tau}) = Q_k \nabla \cdot \boldsymbol{\tau} \quad \forall \boldsymbol{\tau} \in H(\text{div}; \Omega) \cap H^s(\Omega)^d \text{ with } s > 0. \quad (4.6)$$

Remark 4.3. We use $H(\text{div}; \Omega) \cap [H^s(\Omega)]^d$ instead of the choice $\{\boldsymbol{\tau} \in L^p(\Omega)^d \text{ and } \nabla \cdot \boldsymbol{\tau} \in L^2(\Omega)\}$ for $p > 2$ or $W^{1,t}(K)$ for $t > 2d/(d+2)$ in [25, 6, 4] because this choice is more suitable for our analysis.

Theorem 4.4. Let u and $(\boldsymbol{\sigma}_k^m, u_k^m)$ be the solutions of (1.3) and (3.3), respectively. Assume that $u \in H^{1+s}(\Omega)$ with $s > 0$ and that $u|_K \in H^{1+s_K}(K)$ with element-wise defined $s_K > 0$ for all $K \in \mathcal{T}$. Then there exists a constant $C > 0$ independent of the jump of α for both the two- and three-dimension such that

$$\|\alpha^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_k^m)\|_{0,\Omega} \leq \|\alpha^{-1/2}(\boldsymbol{\sigma} - I_k^{rt} \boldsymbol{\sigma})\|_{0,\Omega} \leq C \left(\sum_{K \in \mathcal{T}} h_K^{\min\{s_K, k+1\}} \alpha |\nabla u|_{s_K, K} \right)^{1/2}. \quad (4.7)$$

Proof. The second inequality in (4.7) is a direct consequence of the local approximation property in (4.5). To establish the first inequality in (4.7), denote by

$$\mathbf{E} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_k^m \quad \text{and} \quad e = u - u_k^m$$

the respective errors of the flux and the solution. The commutativity property in (4.6) and the second equations in (3.2) and (3.3) lead to

$$\nabla \cdot (I_k^{rt} \boldsymbol{\sigma}) = Q_k \nabla \cdot \boldsymbol{\sigma} = Q_k f = \nabla \cdot \boldsymbol{\sigma}_k^m.$$

Now, it follows from the first equation in (3.4) and the Cauchy-Schwarz inequality that

$$\begin{aligned} \|\alpha^{-1/2} \mathbf{E}\|_{0,\Omega}^2 &= (\alpha^{-1} \mathbf{E}, \boldsymbol{\sigma} - I_k^{rt} \boldsymbol{\sigma}) + (\alpha^{-1} \mathbf{E}, I_k^{rt} \boldsymbol{\sigma} - \boldsymbol{\sigma}_k^m) \\ &= (\alpha^{-1} \mathbf{E}, \boldsymbol{\sigma} - I_k^{rt} \boldsymbol{\sigma}) + (\nabla \cdot (I_k^{rt} \boldsymbol{\sigma} - \boldsymbol{\sigma}_k^m), e) \\ &= (\alpha^{-1} \mathbf{E}, \boldsymbol{\sigma} - I_k^{rt} \boldsymbol{\sigma}) \leq \|\alpha^{-1/2} \mathbf{E}\|_{0,\Omega} \|\alpha^{-1/2}(\boldsymbol{\sigma} - I_k^{rt} \boldsymbol{\sigma})\|_{0,\Omega}, \end{aligned}$$

which implies the first inequality in (4.7). This completes the proof of the theorem. \square

4.3 discontinuous Galerkin finite element method

Theorem 4.5. Let u be the solution of (1.3) and $u|_K$ be its restriction on $K \in \mathcal{T}$. Assume that $u \in H^{1+s}(\Omega) \cap V^{1+s}(\mathcal{T})$ with $s > 0$ and that $u|_K \in H^{1+s_K}(K)$ with element-wise defined

$s_K > 0$ for all $K \in \mathcal{T}$. Let $u_k^{dg} \in D_k$ be the discontinuous Galerkin finite element approximation in (3.9). Let

$$app_\alpha(f, K) = \begin{cases} \frac{h_K}{\sqrt{\alpha_K}} \|f - f_0\|_{0,K}, & \text{if } 0 < s_K < 1, \\ h_K^{\min\{k, s_K\}} \alpha_K^{1/2} |\nabla u|_{s_K, K}, & \text{if } s_K \geq 1. \end{cases} \quad (4.8)$$

In both the two- and three-dimension, we have the following error estimates:

$$\begin{aligned} \|u - u_k^{dg}\|_{dg} &\leq C \left(\inf_{v \in D_k} \|u - v\|_{dg} + \text{osc}_\alpha(f, \mathcal{T}) \right) \\ &\leq C \left(\sum_{K \in \mathcal{T}} h_K^{2s_K} |\alpha^{1/2} \nabla u|_{s_K, K}^2 + app_\alpha(f, K)^2 \right)^{1/2}, \end{aligned} \quad (4.9)$$

where C is a positive constant independent of the jump of the diffusion coefficient α .

Proof. For any $F \in \mathcal{E}$, it follows from the trace inequality and (3.6) that for all $v \in D_k$

$$\begin{aligned} \sqrt{\alpha_{F,H}/h_F} \llbracket u - v \rrbracket_{0,F} &\leq \sqrt{\alpha_{F,H}/h_F} \left(\|(u - v)|_{K_F^+}\|_{0,F} + \|(u - v)|_{K_F^-}\|_{0,F} \right) \\ &\leq C \sum_{\kappa=-,+} \left(h_{K_F^\kappa}^{-1} \|\alpha^{1/2}(u - v)\|_{0, K_F^\kappa} + \|\alpha^{1/2} \nabla(u - v)\|_{0, K_F^\kappa} \right). \end{aligned}$$

Since $u|_K \in H^{1+s_K}(K)$ with $s_K \geq 1$, then $f|_K = -\alpha \Delta u|_K \in H^{s_K-1}(K)$. It is easy to show that

$$\text{osc}_\alpha(f, K) \leq h_K^{\min\{k, s_K\}} \alpha_K^{1/2} |\nabla u|_{s_K, K}.$$

Now, the second inequality in (4.9) is a direct consequence of the first inequality in (4.9) and the elementwise approximation property of discontinuous piecewise polynomials. By the triangle inequality, we have

$$\|u - u_k^{dg}\|_{dg} \leq \|u - v\|_{dg} + \|u_k^{dg} - v\|_{dg} \quad \forall v \in D_k.$$

To show the validity of the first inequality in (4.9), it suffices to prove that

$$\|u_k^{dg} - v\|_{dg} \leq C (\|u - v\|_{dg} + \text{osc}_\alpha(f, \mathcal{T})) \quad \forall v \in D_k. \quad (4.10)$$

To this end, for any $v \in D_k$, let

$$e = u - v \quad \text{and} \quad e_k = u_k^{dg} - v.$$

It follows from the coercivity in (3.11), the error equation in (3.10), the Cauchy-Schwarz inequality, the fact that $\llbracket u \rrbracket_F = 0$ for all $F \in \mathcal{E}$, and the first inequality in (2.7) that

$$\begin{aligned} C \|e_k\|_{dg}^2 &\leq a_{dg}(e_k, e_k) = a_{dg}(e, e_k) \\ &= (\alpha \nabla_h e, \nabla_h e_k) + \sum_{F \in \mathcal{E}} \int_F \frac{\gamma \alpha_{F,H} \llbracket e \rrbracket \llbracket e_k \rrbracket}{h_F} ds - \sum_{F \in \mathcal{E}} \int_F \{\alpha \nabla e_k \cdot \mathbf{n}\}_w^F \llbracket e \rrbracket ds - \sum_{F \in \mathcal{E}} \int_F \{\alpha \nabla e \cdot \mathbf{n}\}_w^F \llbracket e_k \rrbracket ds \\ &\leq C \left\{ \|e\|_{dg} \|e_k\|_{dg} + \sum_{F \in \mathcal{E}} \|\llbracket e \rrbracket\|_{0,F} \|\{\alpha \nabla e_k \cdot \mathbf{n}\}_w^F\|_{0,F} \right. \\ &\quad \left. + \sum_{F \in \mathcal{E}} h_F^{-1/2} \|\llbracket e_k \rrbracket\|_{0,F} \sum_{\kappa=-,+} w_F^\kappa \alpha_F^\kappa (\|\nabla e\|_{0, K_F^\kappa} + h_K \|\Delta e\|_{0, K_F^\kappa}) \right\} \\ &\equiv C (I_1 + I_2 + I_3). \end{aligned}$$

By the triangle, trace, and inverse inequalities, we have that

$$\|\{\alpha \nabla e_k \cdot \mathbf{n}\}_w^F\|_{0,F} \leq C \sum_{\kappa=-,+} w_F^\kappa h_{K_F^\kappa}^{-1/2} \alpha_{K_F^\kappa}^{1/2} \|\alpha^{1/2} \nabla e_k\|_{0,K_F^\kappa}.$$

With the choice of the weights in (3.8), a simple calculation shows that

$$w_F^\kappa \sqrt{\frac{\alpha_F^\kappa}{\alpha_{F,H}}} \leq \frac{\sqrt{2}}{2} \quad \text{for } \kappa = -, +.$$

Together with the Cauchy-Schwarz inequality, we have

$$\begin{aligned} I_2 &\leq C \sum_{F \in \mathcal{E}} \|\llbracket e \rrbracket\|_{J,F} \sum_{\kappa=-,+} \|\alpha^{1/2} \nabla e_k\|_{0,K_F^\kappa} \leq C \|e\|_{dg} \|e_k\|_{dg} \\ I_3 &\leq C \sum_{F \in \mathcal{E}} \|\llbracket e_k \rrbracket\|_{J,F} \sum_{\kappa=-,+} \left(\|\alpha^{1/2} \nabla e\|_{0,K_F^\kappa} + h_{K_F^\kappa} \|\alpha^{1/2} \Delta e\|_{0,K_F^\kappa} \right) \\ &\leq C \|e_k\|_{dg} \left(\|e\|_{dg} + \left(\sum_{K \in \mathcal{T}} h_K^2 \alpha_K \|\Delta e\|_{0,K}^2 \right)^{1/2} \right). \end{aligned}$$

Combining those inequalities gives that

$$\|e_k\|_{dg} \leq C \left(\|e\|_{dg} + \left(\sum_{K \in \mathcal{T}} h_K^2 \alpha_K \|\Delta e\|_{0,K}^2 \right)^{1/2} \right).$$

Now, (4.10) is a direct consequence of the following efficiency bound (see, e.g, Lemma 5.2 in [7] for the linear case):

$$h_K \alpha_K^{1/2} \|\Delta e\|_{0,K} \leq C \left(\|\alpha^{1/2} \nabla e\|_{0,\omega_K} + \text{osc}_\alpha(f, \omega_K) \right).$$

This completes the proof of the inequality in (4.10) and, hence, the theorem. \square

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